

# FØLNER SEQUENCES IN OPERATOR THEORY AND OPERATOR ALGEBRAS

PERE ARA, FERNANDO LLEDÓ, AND DMITRY V. YAKUBOVICH

**ABSTRACT.** The present article is a review of recent developments concerning the notion of Følner sequences both in operator theory and operator algebras. We also give a new direct proof that any essentially normal operator has an increasing Følner sequence  $\{P_n\}$  of non-zero finite rank projections that strongly converges to  $\mathbb{1}$ . The proof is based on Brown-Douglas-Fillmore theory. We use Følner sequences to analyze the class of finite operators introduced by Williams in 1970. In the second part of this article we examine a procedure of approximating any amenable trace on a unital and separable  $C^*$ -algebra by tracial states  $\text{Tr}(\cdot P_n)/\text{Tr}(P_n)$  corresponding to a Følner sequence and apply this method to improve spectral approximation results due to Arveson and Bédos. The article concludes with the analysis of  $C^*$ -algebras admitting a non-degenerate representation which has a Følner sequence or, equivalently, an amenable trace. We give an abstract characterization of these algebras in terms of unital completely positive maps and define Følner  $C^*$ -algebras as those unital separable  $C^*$ -algebras that satisfy these equivalent conditions. This is analogous to Voiculescu's abstract characterization of quasidiagonal  $C^*$ -algebras.

## 1. INTRODUCTION

In their beginnings the single operator theory and the theory of operator algebras were a common subject and shared many techniques. As an example recall that von Neumann algebras were introduced (as *rings of operators*) in 1929 by von Neumann in his second paper on spectral theory [39]. In recent times, however, each of these theories has developed own elaborated techniques which in many cases remain unknown to experts of the other area. Nevertheless single operator theory and the theory of operator algebras have also had fruitful and important interactions ever since. Brown, Douglas and Fillmore's theory was motivated by the classification of essentially normal operators and ended with the introduction of the *Ext*-group as a fundamental invariant for operator algebras. Finally, Voiculescu's work on quasidiagonality also shows the importance of the dialog between these communities (cf. [26, 49, 50, 48]).

In more recent times operator algebra techniques, in particular exact  $C^*$ -algebras, have also been used to solve Herrero's approximation problem for quasidiagonal operators (cf. [15]). Moreover, operator algebras have shown to be a useful tool in order to address problems in spectral approximation: given a sequence of linear operators  $\{T_n\}_{n \in \mathbb{N}}$  in a complex separable Hilbert space  $\mathcal{H}$  that approximates an operator  $T$  in a suitable sense, a natural question is how do the spectral characteristics of  $T$  (the spectrum, spectral measures, numerical ranges, pseudospectra etc.) relate with those of  $T_n$  as  $n$  grows. (Excellent books that include a large number of examples and references are, e.g., [21, 2]. See also [11, 29] for the application of  $C^*$ -algebra techniques in numerical analysis.) Arveson's seminal series of articles [3, 4, 5] on

---

*Date:* April 10, 2013.

*Key words and phrases.* Følner sequences, non-normal operators, essentially normal operators,  $C^*$ -algebra, amenable trace, spectral approximation.

The first-named author was partially supported by the project MTM2011-28992-C02-01 of the Spanish Ministry of Science and Innovation (MICINN), also supported by FEDER, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The second-named and third-named authors were partly supported by the MICINN and FEDER projects MTM2012-36372-C03-01 and, respectively, MTM2008-06621-C02-01, MTM2011-28149-C02-1. The last two authors also acknowledge the support by the ICMAT Severo Ochoa project SEV-2011-0087.

this topic were directly inspired by Szegő's classical approximation theorem for Toeplitz operators. Among other interesting results, Arveson gave conditions that guarantee that the essential spectrum of a large class of selfadjoint operators  $T$  may be recovered from the sequence of eigenvalues of certain finite dimensional compressions  $T_n$ . These results were then refined by Bédos who systematically applied the concept of Følner sequence of non-zero finite rank projections to spectral approximation problems (see [8, 7, 6] as well as [29, 37]; for a precise definition of Følner sequence and additional results we refer to Section 2). It is stated in Section 7.2 of [29] that SeLegue also considered Szegő-type theorems for Toeplitz operators in the context of  $C^*$ -algebras. Hansen extends some of the mentioned results to the case of unbounded operators (cf. [33, § 7]; see also [34] for recent developments in the non-selfadjoint case). Brown shows in [16] that abstract results in  $C^*$ -algebra theory can be applied to compute spectra of important operators in mathematical physics like almost Mathieu operators or periodic magnetic Schrödinger operators on graphs.

In the last two decades, the relation between spectral approximation problems and Følner sequences for non-selfadjoint and non-normal operators has been also explored, see for instance [52, 45, 12, 43].

The aim of this article is to present in a single publication recent operator theoretic and operators algebraic results that involve the notion of Følner sequences for operator. Følner sequences were introduced in the context of operator algebras by Connes in Section V of his seminal paper [22] (see also [23, Section 2]). This notion is an algebraic analogue of Følner's characterization of amenable discrete groups and was used by Connes as an essential tool in the classification of injective type  $II_1$  factors. Part of the material of this paper is taken from [1, 38]. There is though a new and complete proof that any essentially normal operator has a proper Følner sequence (cf. Subsection 3.1) which, in our opinion, is interesting in its own right. The proof is based on the absorbing property for direct sums stated in Proposition 2.3 and pure operator theoretic arguments including Brown-Douglas-Fillmore theory.

In this article we will present (with the exception of Subsection 3.1) only short proofs that improve the comprehension of the statement or that contain useful techniques. For more difficult and elaborate arguments we will refer to the original publications. Section 3 is completed with the analysis of the relations between the class of finite operators (introduced by Williams in [53]) and the notion of Følner sequence. It is shown that Følner sequences for operators provide a very useful and natural tool to analyze this class of operators. In the last section we will study the role of Følner sequences in operator algebras. First we review the relation between Følner sequences for a unital and separable  $C^*$ -algebra  $\mathcal{A}$  and amenable traces. In particular, we present an approximation procedure for amenable traces in terms of Følner sequences of projections [41, Theorem 6.1] (see also [17, Theorem 6.2.7]). We apply this method in Theorem 4.2 to extend a spectral approximation result for scalar spectral measures in the spirit of Arveson and Bédos. In Subsection 4.2 we give finally an abstract characterization of unital separable  $C^*$ -algebras  $\mathcal{A}$  admitting a non-degenerate representation  $\pi$  on a Hilbert space such that there is a Følner sequence for  $\pi(\mathcal{A})$  or, equivalently, such that  $\pi(\mathcal{A})$  has an amenable trace (see Theorem 4.7). We conclude with a brief discussion of  $C^*$ -algebras that can also be related to a given Følner sequence and that appear naturally in the context of spectral approximation problems. In the last section we summarize some of the main relations and differences in the analysis of Følner sequences for single operators and for abstract  $C^*$ -algebras.

**Notation:** We will denote by  $\mathcal{L}(\mathcal{H})$  the  $C^*$ -algebra of bounded and linear operators on the complex separable Hilbert space  $\mathcal{H}$ , and by  $\mathcal{K}(\mathcal{H})$  the ideal of compact operators on  $\mathcal{H}$ . Next,  $\mathcal{P}_{\text{fin}}(\mathcal{H})$  is the set of all non-zero finite rank orthogonal projections on  $\mathcal{H}$  and  $[A, B] := AB - BA$  stands for the commutator of two operators  $A, B \in \mathcal{L}(\mathcal{H})$ . We denote by  $\text{Tr}(\cdot)$  the standard trace on  $\mathcal{L}(\mathcal{H})$  and by  $\text{tr}(\cdot)$  the unique tracial state on a matrix algebra  $M_n(\mathbb{C})$ ,  $n \in \mathbb{N}$ .

## 2. BASIC PROPERTIES OF FØLNER SEQUENCES FOR OPERATORS

The notion of Følner sequences for operators has its origins in group theory. Recall that a discrete countable group  $\Gamma$  is said to be *amenable* if it has an invariant mean, i.e., there is a positive linear functional  $\psi$  on the von Neumann algebra<sup>1</sup>  $\ell^\infty(\Gamma)$  with norm one such that

$$\psi(\gamma f) = \psi(f), \quad \gamma \in \Gamma, \quad f \in \ell^\infty(\Gamma),$$

where  $(\gamma f)(\gamma_0) := f(\gamma^{-1}\gamma_0)$ . Abelian groups, finite groups and their extensions are amenable. A Følner sequence for the group  $\Gamma$  is a sequence of non-empty finite subsets  $\Gamma_i \subset \Gamma$  that satisfy

$$(2.1) \quad \lim_i \frac{|(\gamma \Gamma_i) \Delta \Gamma_i|}{|\Gamma_i|} = 0 \quad \text{for all } \gamma \in \Gamma,$$

where  $\Delta$  denotes the symmetric difference and  $|X|$  is the cardinality of a set  $X$ . Then,  $\Gamma$  has a Følner sequence if and only if  $\Gamma$  is amenable (see, e.g., Chapter 4 in [42]). An analysis of different properties of approximability of a group by finite groups and their relation to amenability has been undertaken in the review [47].

The counterpart of the preceding definition in the context of operators is given next. First we need to recall that if  $T \in \mathcal{L}(\mathcal{H})$ , then  $\|T\|_p$ ,  $p = 1, 2, \dots$ , is its norm in the Schatten-von Neumann class.

**Definition 2.1.** Let  $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$  be a set of operators. A sequence of non-zero finite rank orthogonal projections  $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_{\text{fin}}(\mathcal{H})$  on  $\mathcal{H}$  is called a *Følner sequence for  $\mathcal{T}$*  if

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{\|TP_n - P_nT\|_2}{\|P_n\|_2} = 0, \quad T \in \mathcal{T}.$$

If the Følner sequence  $\{P_n\}_{n \in \mathbb{N}}$  satisfies, in addition, that it is increasing and converges strongly to  $\mathbb{1}$ , then we say it is a *proper Følner sequence for  $\mathcal{T}$* .

The existence of a Følner sequence has already important structural consequences, see for instance Proposition 2.5 and Corollary 3.9 below. Notice, however, that proper Følner sequences are important in the context of spectral approximation in the spirit of works [52, 45, 43] and others.

In the preceding definition we have not specified any structure on the set of operators  $\mathcal{T}$ . Typically,  $\mathcal{T}$  will be a single operator or a concrete  $C^*$ -algebra, realized in a Hilbert space  $\mathcal{H}$ . The next result collects some immediate consequences of the definition of a Følner sequence for operators. Part (ii) is shown in Lemma 1 of [6] (see also [38, Proposition 2.1]).

**Proposition 2.2.** Let  $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$  be a set of operators and  $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_{\text{fin}}(\mathcal{H})$  a sequence of non-zero finite rank orthogonal projections. Then we have

- (i)  $\{P_n\}_{n \in \mathbb{N}}$  is a Følner sequence for  $\mathcal{T}$  if and only if it is a Følner sequence for  $C^*(\mathcal{T}, \mathbb{1})$ , where  $C^*(\cdot)$  is the  $C^*$ -algebra generated by its argument. Moreover,  $\{P_n\}_{n \in \mathbb{N}}$  is a proper Følner sequence for  $\mathcal{T}$  if and only if it is a proper Følner sequence for

$$C^*(\mathcal{T}, \mathcal{K}(\mathcal{H}), \mathbb{1}).$$

- (ii)  $\{P_n\}_{n \in \mathbb{N}}$  is a Følner sequence for  $\mathcal{T}$  if and only if the following condition holds:

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{\|TP_n - P_nT\|_1}{\|P_n\|_1} = 0, \quad T \in \mathcal{T}.$$

If  $\mathcal{T}$  is a self-adjoint set (i.e.,  $\mathcal{T}^* = \mathcal{T}$ ), then  $\{P_n\}_{n \in \mathbb{N}}$  is a Følner sequence for  $\mathcal{T}$  if and only if for all  $T \in \mathcal{T}$ :

$$(2.4) \quad \lim_{n \rightarrow \infty} \frac{\|(I - P_n)TP_n\|_p}{\|P_n\|_p} = 0, \quad p \in \{1, 2\}.$$

<sup>1</sup>We identify here each  $f \in \ell^\infty(\Gamma)$  with the multiplication operator with  $f$  on the Hilbert space  $\ell_2(\Gamma)$ .

- (iii) Let  $T \in \mathcal{L}(\mathcal{H})$  and  $\{Q_n\}_n \subset \mathcal{P}_{fin}(\mathcal{H})$  be such that the sequence  $\{\dim Q_n\}$  is unbounded and

$$\lim_{n \rightarrow \infty} \frac{\|TQ_n - Q_nT\|_2}{\|Q_n\|_2} = 0.$$

Then there exists a proper Følner sequence for  $T$ .

Proper Følner sequences for operators have the following characteristic absorbing property for direct sums:

**Proposition 2.3.** *Let  $\mathcal{H}$  and  $\mathcal{H}'$  be separable Hilbert spaces with  $\dim \mathcal{H} = \infty$ . If  $T$  has a proper Følner sequence, then  $T \oplus X \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')$  has a proper Følner sequence for any  $X \in \mathcal{L}(\mathcal{H}')$ .*

*Proof.* Let  $\{P_n\}_{n \in \mathbb{N}}$  be a proper Følner sequence for  $T$  and since the sequence of projections is increasing we may assume that  $\dim P_n \mathcal{H} \geq n^2$ . Let  $\{e_i \mid i \in \mathbb{N}\}$  be an orthonormal basis of  $\mathcal{H}'$  and denote by  $Q_n$  the orthogonal projection onto  $\text{span}\{e_1, \dots, e_n\} \subset \mathcal{H}'$ . Then the following calculation shows that  $\{P_n \oplus Q_n\}_n$  is a proper Følner sequence for  $T \oplus X$ ,  $X \in \mathcal{L}(\mathcal{H}')$ :

$$\begin{aligned} \frac{\|[T \oplus X, P_n \oplus Q_n]\|_2^2}{\|P_n \oplus Q_n\|_2^2} &= \frac{\|[T, P_n]\|_2^2 + \|[X, Q_n]\|_2^2}{\|P_n\|_2^2 + n} \\ &\leq \frac{\|[T, P_n]\|_2^2}{\|P_n\|_2^2} + \frac{4\|Q_n\|_2^2 \|X\|^2}{n^2 + n} \\ &= \frac{\|[T, P_n]\|_2^2}{\|P_n\|_2^2} + 4\|X\| \frac{n}{n^2 + n} \xrightarrow{n \rightarrow \infty} 0. \quad \square \end{aligned}$$

Next, we mention some first operator algebraic consequences related to the existence of Følner sequences. For this we need to recall the following notion:

**Definition 2.4.** A state  $\tau$  on the unital  $C^*$ -algebra  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$  (i.e., a positive and normalized linear functional on  $\mathcal{A}$ ) is called an *amenable trace* if there exists a state  $\psi$  on  $\mathcal{L}(\mathcal{H})$  such that  $\psi \upharpoonright \mathcal{A} = \tau$  and

$$\psi(XA) = \psi(AX), \quad X \in \mathcal{L}(\mathcal{H}), \quad A \in \mathcal{A}.$$

The state  $\psi$  is also referred in the literature as a *hypertrace* for  $\mathcal{A}$ .

Note that an amenable trace is really a trace on  $\mathcal{A}$  (i.e.,  $\tau(AB) = \tau(BA)$ ,  $A, B \in \mathcal{A}$ ). We also refer to [13, 41] for a thorough description of the relations of amenable traces and Følner sequences to other important areas like, e.g., Connes' embedding problem. Hypertraces are the algebraic analogue of the invariant mean on groups mentioned at the beginning of this section. Later we will need the following standard result. (See [22, 23] for the original statement and more results in the context of operator algebras; see also [7, 1] for additional results in the context of  $C^*$ -algebras related to the existence of a hypertrace.)

**Proposition 2.5.** *Let  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$  be a separable unital  $C^*$ -algebra. Then  $\mathcal{A}$  has a Følner sequence if and only if  $\mathcal{A}$  has an amenable trace.*

Finally, we also mention the following useful results in the context of single operator theory. We need to introduce first the following definition.

**Definition 2.6.** We say that  $T \in \mathcal{L}(\mathcal{H})$  is *finite block reducible* if  $T$  has a non-trivial finite-dimensional reducing subspace, i.e., there is an orthogonal decomposition  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  which reduces  $T$  and where  $\mathcal{H}_0$  is finite dimensional and non-zero.

The following two propositions are technical and we refer to Section 3 in [38] for a complete proof.

**Proposition 2.7.** *Let  $T = T_0 \oplus \tilde{T}$  on  $\mathcal{H} = \mathcal{H}_0 \oplus \tilde{\mathcal{H}}$ , where  $\dim \mathcal{H}_0 < \infty$ . Then,  $T$  has a proper Følner sequence if and only if  $\tilde{T}$  has a proper Følner sequence.*

Note that in the reverse implication of Proposition 2.5 the sequence of projections does not have to be a proper Følner sequence in the sense of Definition 2.1. In fact, one can easily construct the following counterexample: consider a finite block reducible operator  $T = T_0 \oplus T_1$  on the Hilbert space  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ , with  $1 \leq \dim \mathcal{H}_0 < \infty$  and where  $T_1$  has no Følner sequence (examples of these type of operators will be given in Section 3.3). Then, one can show that  $C^*(T, \mathbb{1})$  has a hypertrace (see Williams theorem in Subsection 3.2) and by Proposition 2.5 it has a Følner sequence also. The obvious choice of Følner sequence is the constant sequence  $P_n = \mathbb{1}_{\mathcal{H}_0} \oplus 0$ ,  $n \in \mathbb{N}$ , which trivially satisfies (2.2) for  $T$ . But  $T$  cannot have a proper Følner sequence, because  $T_1$  has no Følner sequence by Proposition 2.7.

The following proposition clarifies the relation between Følner sequences and proper Følner sequences in the context of operator theory. In a sense the difference between Følner sequence and proper Følner sequence can only appear if the operator is finite block reducible.

**Proposition 2.8.** *Let  $T \in \mathcal{L}(\mathcal{H})$  and suppose that  $TP - PT \neq 0$  for all  $P \in \mathcal{P}_{fin}(\mathcal{H})$ . If there is a Følner sequence of projections  $\{P_n\}_n \subset \mathcal{P}_{fin}(\mathcal{H})$  of a constant rank, then  $T$  has a proper Følner sequence.*

### 3. FØLNER SEQUENCES IN OPERATOR THEORY

Using a classical result by Berg that states that any normal operator can be expressed as a sum of a diagonal operator and a compact operator (cf. [26, Section II.4]) it is immediate that any normal operator has a proper Følner sequence. In the next subsection we will address the question of existence of proper Følner sequences for an important class of non-normal operators. We will also explore the structure of operators that have no proper Følner sequence. Finally we will show the strong link between the class of finite operators (in the sense of Williams [53]) and the notion of Følner sequence.

**3.1. Essentially normal operators.** In this subsection we give a proof of the fact that any essentially normal operator has a proper Følner sequence. The proof below is from an earlier version of [38]. In this reference we present a stronger statement, namely that any essentially hyponormal operator has a proper Følner sequence by using different techniques (see Theorem 5.1 in [38])<sup>2</sup>.

Nevertheless in our opinion the present direct proof is interesting in itself and the reasoning is completely different from that in [38]. The proof below is based on the absorbing property for direct sums given in Proposition 2.3 and pure operator theoretic arguments including Brown-Douglas-Fillmore theory.

We begin showing that the unilateral shift  $S$  has a canonical proper Følner sequence. In fact, define  $S$  on  $\mathcal{H} := \ell^2(\mathbb{N}_0)$  by  $Se_i := e_{i+1}$ , where  $\{e_i \mid i = 0, 1, 2, \dots\}$  is the canonical basis of  $\mathcal{H}$  and consider for any  $n$  the orthogonal projection  $P_n$  onto  $\text{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$ . Then

$$\|[P_n, S]\|_2^2 = \sum_{i=1}^{\infty} \|[P_n, S]e_i\|^2 = \|e_{n+1}\|^2 = 1$$

and

$$\frac{\|[P_n, S]\|_2}{\|P_n\|_2} = \frac{1}{\sqrt{n+1}} \xrightarrow{n \rightarrow \infty} 0.$$

Next we recall some definitions and facts concerning essentially normal operators. Details and additional references can be found, e.g., in [19, 28]; see also [27] for an excellent brief up-to-date account of essential normality and the Brown–Douglas–Fillmore theory. An operator

<sup>2</sup>An operator  $T \in \mathcal{L}(\mathcal{H})$  is called *hyponormal* if its self-commutator  $[T^*, T]$  is nonnegative.  $T$  is called *essentially hyponormal* if the image in the Calkin algebra  $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$  of  $[T^*, T]$  is a nonnegative element. Any essentially normal operator is essentially hyponormal (see, e.g., [24, Chapter 4] or the review [51] for additional results).

$T \in \mathcal{L}(\mathcal{H})$  is called *essentially normal* if its self-commutator is a compact operator, i.e., if  $[T, T^*] \in \mathcal{K}(\mathcal{H})$ . If  $\rho$  is the quotient map from  $\mathcal{L}(\mathcal{H})$  onto the Calkin algebra  $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ , then  $T$  is essentially normal if and only if  $\rho(T)$  is normal in the Calkin algebra. The unilateral shift  $S$  mentioned above is a standard example of an essentially normal operator, since its self-commutator is a rank 1 projection. We recall that an operator  $F \in \mathcal{L}(\mathcal{H})$  is called *Fredholm* if its range  $\text{ran } F$  is closed and both  $\ker F$  and  $(\text{ran } F)^\perp$  are finite dimensional. The index of a Fredholm operator  $F$  is defined as

$$\text{ind}(F) = \dim \ker F - \dim(\text{ran } F)^\perp .$$

The *essential spectrum* of an operator  $T$  is

$$\sigma_{\text{ess}}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda \mathbb{1} \text{ is not Fredholm}\} .$$

If  $F$  is Fredholm and  $K$  is compact, then  $F + K$  is Fredholm and  $\text{ind}(F + K) = \text{ind}(F)$ . Finally,  $F \in \mathcal{L}(\mathcal{H})$  is Fredholm if and only if  $\rho(F)$  is invertible in the Calkin algebra. Therefore, the essential spectrum of any  $T \in \mathcal{L}(\mathcal{H})$  coincides with the spectrum of  $\rho(T)$ . We refer to Section I.8 in [44] for an accessible exposition of Fredholm operators.

We will need later the following standard facts:

**Proposition 3.1.** *Let  $\{T_n\}_{n \in \mathbb{N}}$  be a sequence of bounded operators in  $\mathcal{L}(\mathcal{H}_n)$ .*

- (i) *Assume  $\sup_n \{\|T_n\|\} < \infty$  and define the bounded operator  $\hat{T} = \oplus_n T_n$  on  $\oplus_n \mathcal{H}_n$ . Then,  $\hat{T}$  is invertible if and only if each  $T_n$  is invertible and*

$$\sup_n \{\|T_n^{-1}\|\} < \infty .$$

- (ii)  $\sigma_{\text{ess}}(T_1 \oplus T_2) = \sigma_{\text{ess}}(T_1) \cup \sigma_{\text{ess}}(T_2)$ .
- (iii) *If  $T_1, T_2$  are Fredholm operators, then  $\text{ind}(T_1 \oplus T_2) = \text{ind}(T_1) + \text{ind}(T_2)$ .*

The proof of the main result of this subsection is based on the existence of operators having specific spectral properties. In what follows, for a subset  $\Omega$  of the complex plane, we denote by  $\Omega^{\text{cl}}$  its closure and put

$$\overline{\Omega} = \{\bar{z} \mid z \in \Omega\} .$$

We will use the space  $R^2(\Omega)$ , defined as the closure in  $L^2(\Omega)$  (with the Lebesgue measure) of the set of rational functions with poles off  $\Omega^{\text{cl}}$  (see, e.g., Chapter 1 in [24]).

We need to recall here also some other standard notions in operator theory. An operator  $T \in \mathcal{L}(\mathcal{H})$  is called *finitely multicyclic* if there are finitely many vectors  $g_1, \dots, g_m \in \mathcal{H}$  such that the span of the set

$$\{u(T)g_i \mid 1 \leq i \leq m, \ u \text{ rational function with poles off } \sigma(T)\}$$

is dense in  $\mathcal{H}$ . The vectors  $g_1, \dots, g_m$  are called a *cyclic set of vectors*. If  $T$  is finitely multicyclic and  $m$  is the smallest number of cyclic vectors, then  $T$  is called  *$m$ -multicyclic*.

For the reader's convenience, we recall the following classical result due to Berger and Shaw and which we will use several times. For details we refer to the original article [9] or to Section IV.2 in [24].

**Theorem 3.2** (Berger-Shaw). *Suppose  $T$  is an  $m$ -multicyclic hyponormal operator. Then its self-commutator  $[T^*, T]$  is of trace class and the canonical trace satisfies*

$$\text{Tr}([T^*, T]) \leq \frac{m}{\pi} \text{area}(\sigma(T)) ,$$

where  $\sigma(T)$  denotes the spectrum of  $T$ .

**Lemma 3.3.** *Let  $\Omega$  be an open, bounded and connected subset of  $\mathbb{C}$ . Then, the multiplication operator on  $R^2(\Omega)$  given by*

$$(M_\Omega f)(z) := z f(z) , \quad f \in R^2(\Omega) ,$$

satisfies the following properties:

- (i)  $\sigma(M_\Omega) = \Omega^{cl}$  and  $\|M_\Omega\| = \max_{z \in \Omega^{cl}} \{|z|\}$ .
- (ii)  $\sigma_{\text{ess}}(M_\Omega) \subset \partial\Omega$  and

$$\text{ind}(M_\Omega - \lambda \mathbb{1}) = \begin{cases} 0, & \lambda \notin \Omega^{cl} \\ -1, & \lambda \in \Omega. \end{cases}$$

- (iii)  $\|(M_\Omega - \lambda)^{-1}\| = (\text{dist}(\lambda, \Omega^{cl}))^{-1}$ ,  $\lambda \notin \Omega^{cl}$ .
- (iv)  $M_\Omega$  is a hyponormal operator.<sup>3</sup>
- (v) The self-commutator  $[M_\Omega^*, M_\Omega]$  is a trace-class operator and

$$(3.1) \quad \text{Tr}([M_\Omega^*, M_\Omega]) \leq \frac{1}{\pi} \text{area}(\Omega).$$

*Proof.* It is a standard fact that  $R^2(\Omega)$  consists of analytic functions on  $\Omega$  and that for any  $\lambda \in \Omega$ , the evaluation functional  $f \mapsto f(\lambda)$  is bounded on  $R^2(\Omega)$  (see, e.g., Section II.7 in [24]). Parts (i) and (iii) follow from standard properties of the multiplication operator. (Note that for  $\lambda \notin \Omega^{cl}$  the function  $(z - \lambda)^{-1}$  is bounded and analytic in  $\Omega$ .) To prove (ii), it suffices to observe that  $\ker(M_\Omega - \lambda) = \{0\}$  for any  $\lambda \in \mathbb{C}$ , that  $\text{Ran}(M_\Omega - \lambda) = R^2(\Omega)$  for  $\lambda \notin \Omega^{cl}$  and that

$$\text{Ran}(M_\Omega - \lambda) = \{f \in R^2(\Omega) \mid f(\lambda) = 0\} \quad \text{for } \lambda \in \Omega.$$

This gives the formula for the index stated above.

To prove (iv) note that  $M_\Omega^* f = Q_R(\bar{z} f)$ ,  $f \in R^2(\Omega)$ , where  $Q_R$  denotes the orthogonal projection from  $L^2(\Omega)$  onto  $R^2(\Omega)$ . Therefore  $\|M_\Omega^* f\| \leq \|M_\Omega f\|$ ,  $f \in R^2(\Omega)$ , which implies that  $M_\Omega$  is a hyponormal. Finally, by the definition of  $R^2(\Omega)$ , the constant function 1 is cyclic for  $M_\Omega$ , so that  $M_\Omega$  is 1-multicyclic. Hence we can apply Berger-Shaw Theorem to conclude that  $[M_\Omega^*, M_\Omega]$  is a trace-class operator and that the inequality stated above holds.  $\square$

**Definition 3.4.** Let  $T, R \in \mathcal{L}(\mathcal{H})$  be essentially normal operators. We say that  $T$  and  $R$  have the same spectral picture the following two conditions hold:

- (i)  $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(R) =: X$
- (ii)  $\text{ind}(T - \lambda \mathbb{1}) = \text{ind}(R - \lambda \mathbb{1})$ ,  $\lambda \notin X$ .

**Theorem 3.5.** Any essentially normal operator  $T \in \mathcal{L}(\mathcal{H})$  has a proper Følner sequence.

*Proof.* (i) The first step of the proof uses the following classical result of the Brown-Douglas-Fillmore theory (see [18, Section V] or [19, Theorem 11.1]). Let  $T, R \in \mathcal{L}(\mathcal{H})$  be essentially normal, then we have that  $T = U(R + K)U^*$  for some compact operator  $K$  and some unitary  $U$  if and only if operators  $T$  and  $R$  have the same spectral picture (cf. Definition 3.4). Therefore to prove that  $T$  has a proper Følner sequence it will be enough to construct an essentially normal operator  $R$  with a proper Følner sequence and having the same spectral picture as  $T$ . Indeed, if  $\{P_n\}_n$  is a proper Følner sequence for  $R$ , then by Proposition 2.2 (i) it is also a proper Følner sequence for  $R + K$  for any compact operator  $K$  and therefore  $\hat{P}_n := UP_nU^*$  is a proper Følner sequence for  $T = U(R + K)U^*$ .

(ii) Given the essentially normal operator  $T$ , the construction of an essentially normal operator  $R$  with the same spectral picture as  $T$  and having a proper Følner sequence goes as follows. The set  $X := \sigma_{\text{ess}}(T)$  is a closed and bounded subset of  $\mathbb{C}$ , so that we consider its decomposition

$$\mathbb{C} \setminus X := \bigcup_{j \in J} \Omega_j$$

into open, connected and disjoint sets; here  $J \subset \mathbb{N}$  is a set of indices. The index function  $\bigcup_j \Omega_j \ni \lambda \mapsto \text{ind}(T - \lambda \mathbb{1})$  is continuous and therefore constant on each connected component  $\Omega_j$ .

---

<sup>3</sup>In fact a stronger property holds:  $M_\Omega$  is a subnormal operator (i.e., the restriction of a normal operator to an invariant subspace); see [24].

We denote for  $\lambda \in \Omega_j$  the index by  $n_j := \text{ind}(T - \lambda \mathbb{1}) \in \mathbb{Z}$ , and put

$$J_- := \{j \in J \mid n_j < 0\}, \quad J_+ := \{j \in J \mid n_j > 0\}, \quad J_\cup = J_- \cup J_+.$$

These sets of indices may be finite or infinite.

To construct  $R$ , first take any normal operator  $N$  on an infinite dimensional Hilbert space  $\mathcal{K}$  such that  $\sigma_{\text{ess}}(N) = X$ . (A concrete example can be constructed as follows: put  $\mathcal{H} = \ell^2(\mathbb{N})$  and let  $\{d_n\}_{n \in \mathbb{N}}$  be a dense sequence of points in  $X$ . Any isolated point in  $X$  is repeated infinitely many times. Then the diagonal operator  $N := \text{diag}(\{d_n\}_n)$  is normal and  $\sigma_{\text{ess}}(N) = X$ .) Since  $N$  is normal we have  $\text{ind}(N - \lambda \mathbb{1}) = 0$ ,  $\lambda \notin X$ .

Second, for any bounded  $\Omega_j$ ,  $j \in J_+$  (i.e.,  $n_j > 0$ ) we consider the operator  $M_j := M_{\Omega_j}$  on  $R^2(\Omega_j)$  as in Lemma 3.3 and that satisfies the properties (i)-(iv). If  $j \in J_-$ , then we put  $M_j := M_{\overline{\Omega_j}}$  on  $R^2(\overline{\Omega_j})$ . Define the Hilbert spaces  $\mathcal{K}_j := \oplus^{n_j} R^2(\Omega_j)$  for  $n_j > 0$  and  $\mathcal{K}_j := \oplus^{|n_j|} R^2(\overline{\Omega_j})$  for  $n_j < 0$ . Next, we construct on  $\mathcal{K}_j$  the operator

$$S_j := \begin{cases} \bigoplus_1^{|n_j|} M_j^*, & \text{if } n_j < 0, \\ \bigoplus_1^{n_j} M_j, & \text{if } n_j > 0. \end{cases}$$

From Proposition 3.1 (iii) and Lemma 3.3 (ii) we have  $\text{ind}(S_j - \lambda \mathbb{1}) = n_j$  for any  $\lambda \in \Omega_j$ . Then we consider the operator

$$\widehat{S} := \left( \bigoplus_{j \in J_\cup} S_j \right) \quad \text{on} \quad \widehat{\mathcal{K}} := \bigoplus_{j \in J_\cup} \mathcal{K}_j$$

and, finally, we put

$$R := N \oplus \widehat{S} \in \mathcal{L}(\mathcal{K} \oplus \widehat{\mathcal{K}}).$$

(iii) The last part of the proof consists in showing that  $R$  satisfies all the required properties.

Since  $N$  is normal it has a proper Følner sequence. By the absorbing property of proper Følner sequences for direct sums stated in Proposition 2.3 we conclude that  $R$  has a proper Følner sequence too.

Next we show that  $R$  has the same spectral picture as the given operator  $T$ . For this purpose we prove first that  $\sigma_{\text{ess}}(\widehat{S}) \subset X$  and that for  $\lambda \in \Omega_j$ ,  $\text{ind}(\widehat{S} - \lambda \mathbb{1}) = n_j$ . Assume that  $\lambda \notin X$ . Then  $\lambda \in \Omega_k$  for some index  $k \in J$ . If  $k \notin J_\cup$ , put  $d := \inf_{j \in J_\cup} \left\{ \text{dist}(\lambda, \Omega_j^{\text{cl}}) \right\}$ . In this case  $d > 0$ . From Lemma 3.3 (iii) we obtain

$$\|(S_j - \lambda \mathbb{1})^{-1}\| = \frac{1}{\text{dist}(\lambda, \Omega_j^{\text{cl}})} \leq \frac{1}{d}, \quad j \in J_\cup.$$

We conclude that the operator  $\widehat{S} - \lambda \mathbb{1}$  is invertible (recall Proposition 3.1 (i)), hence it is Fredholm of index 0 and  $\lambda \notin \sigma_{\text{ess}}(\widehat{S})$ .

Now consider the case when  $\lambda \in \Omega_k$ , where  $k \in J_\cup$ . Then we may consider the decomposition

$$\widehat{S} - \lambda \mathbb{1} = (S_k - \lambda \mathbb{1}) \oplus \left( \bigoplus_{j \neq k} S_j - \lambda \mathbb{1} \right).$$

The same argument as before shows that  $\bigoplus_{j \neq k} (S_j - \lambda \mathbb{1})$  is invertible, hence Fredholm of index 0. By construction of  $S_k$  (see Lemma 3.3 (ii)) and by Proposition 3.1 (iii) we conclude that  $\lambda \notin \sigma_{\text{ess}}(\widehat{S})$  and that  $\text{ind}(\widehat{S} - \lambda \mathbb{1}) = n_k$ , for any  $\lambda \in \Omega_k$ . Therefore we have that  $\sigma_{\text{ess}}(\widehat{S}) \subset X$ .

From the properties of the normal operator  $N$  constructed in step (ii), we have  $\sigma_{\text{ess}}(N) = X$ . Using now Proposition 3.1 (ii) we conclude that

$$\sigma_{\text{ess}}(R) = \sigma_{\text{ess}}(N) \cup \sigma_{\text{ess}}(\widehat{S}) = X.$$



Moreover, we have for any  $\lambda \in \Omega_j$

$$\text{ind}(R - \lambda \mathbb{1}) = 0 + n_j = \text{ind}(T - \lambda \mathbb{1}) ,$$

and we have shown that  $T$  and  $R$  have the same spectral picture.

Finally, we still have to show that  $R$  is essentially normal, i.e., that the self-commutator of  $R$  is compact. For this note that

$$[R^*, R] = 0 \oplus [\widehat{S}^*, \widehat{S}] .$$

We need to consider two cases: if the index set  $J_U$  is finite, then by Lemma 3.3 (v) the operator  $\widehat{S}$  is trace class, hence  $R$  is essentially normal. Note that  $\bigcup_{j \in J_U} \Omega_j$  is bounded. Therefore, if the set  $J_U$  has infinite cardinality, then we have, in addition,

$$(3.2) \quad \lim_{J_U \ni j \rightarrow \infty} \text{area}(\Omega_j) = 0 .$$

Consider the partial direct sum  $\widehat{S}_N := \bigoplus_{j \in J_U, j \leq N} S_j$ . Applying again Lemma 3.3 (v) we get

$$\begin{aligned} \left\| [\widehat{S}^*, \widehat{S}] - [\widehat{S}_N^*, \widehat{S}_N] \right\| &= \left\| \bigoplus_{j \in J_U, j > N} [S_j^*, S_j] \right\| = \sup_{j \in J_U, j > N} \left\| [M_j^*, M_j] \right\| \\ &\leq \frac{1}{\pi} \sup_{j \in J_U, j > N} \text{area}(\Omega_j) \rightarrow 0 \quad \text{as } N \rightarrow \infty \end{aligned}$$

(see Eq. (3.2)). Since  $[\widehat{S}_N^*, \widehat{S}_N]$  is a trace-class operator, it follows that the self-commutator  $[R^*, R]$  can be approximated in norm by trace-class operators, hence it is compact and we conclude that  $R$  is essentially normal.  $\square$

**Corollary 3.6.** *If  $T \in \mathcal{L}(\mathcal{H})$  is an  $m$ -multicyclic hyponormal operator, then  $T$  has a proper Følner sequence.*

*Proof.* By Berger-Shaw Theorem it follows that the self-commutator  $[T^*, T]$  is trace-class and, therefore,  $T$  is essentially normal and the assertion follows from Theorem 3.5.  $\square$

We conclude this subsection mentioning that any quasinormal operator (i.e., any operator  $Q$  that commutes with  $Q^*Q$ ) has a proper Følner sequence. Recall also that an operator  $T$  on  $\mathcal{H}$  is called subnormal if there is a normal operator  $N$  acting on a Hilbert space  $\widetilde{\mathcal{H}}$  containing  $\mathcal{H}$  such that  $\mathcal{H}$  is invariant for  $N$  and  $T$  is the restriction of  $N$  to  $\mathcal{H}$ . It can also be shown that any subnormal operator has a proper Følner sequence. See [38] for details and also Chapter II in [24] for the relations between these classes of operators.

**3.2. Finite operators.** In this subsection we study the class of finite operators introduced by Williams in [53] and their relation to proper Følner sequences. (See also [36].)

We begin recalling the main definition and known results.

**Definition 3.7.**  $T \in \mathcal{L}(\mathcal{H})$  is called a *finite operator* if

$$0 \in \left( W([T, X]) \right)^{\text{cl}} \quad \text{for all } X \in \mathcal{L}(\mathcal{H}) ,$$

where  $W(T)$  denotes the numerical range of the operator  $T$ , i.e.,

$$W(T) = \{ \langle Tx, x \rangle \mid x \in \mathcal{H} , \quad \|x\| = 1 \} ,$$

and where the  $(\cdot)^{\text{cl}}$  means the closure of the corresponding subset in  $\mathbb{C}$ .

We collect in the following theorem some standard results due to Williams about the class of finite operators (cf. [53]).

**Theorem 3.8** (Williams). *An operator  $T \in \mathcal{L}(\mathcal{H})$  is finite if and only if  $C^*(T, \mathbb{1})$  has an amenable trace. The class of finite operators is closed in the operator norm and contains all finite block reducible operators.*

It follows that the norm closure of the set of all finite block reducible operators is contained in the class of finite operators. Combining Williams' Theorem with Proposition 2.5, we get the following fact.

**Corollary 3.9.** *For any operator  $T \in \mathcal{L}(H)$ , the following properties are equivalent:*

- (i)  $T$  is finite;
- (ii)  $T$  has a Følner sequence;
- (iii)  $C^*(T, \mathbb{1})$  has an amenable trace.

The next result shows the strong link between finite operators and proper Følner sequences. We include the proof, because it is short and illustrative (cf. [38, Theorem 4.1]).

**Theorem 3.10.** *Let  $T \in \mathcal{L}(\mathcal{H})$ . Then,  $T$  is a finite operator if and only if  $T$  is finite block reducible or  $T$  has a proper Følner sequence.*

*Proof.* (i) If  $T$  is finite block reducible, then  $T$  is a finite operator (cf. [53]). Moreover, if  $T$  has a proper Følner sequence, then the  $C^*$ -algebra  $C^*(T, \mathbb{1})$  has the same proper Følner sequence and, by Proposition 2.5, it also has an amenable trace. Then, by Williams' theorem (see also Theorem 4 in [53]) we conclude that  $T$  is finite.

(ii) To prove the other implication, assume  $T$  is a finite operator. We consider several cases. If there exists a (non-zero)  $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$  such that  $[T, P] = 0$ , then  $T$  is finite block reducible. Consider next the situation where  $[T, P] \neq 0$  for all  $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ . Since  $T$  is finite we can use Williams' Theorem to conclude that  $C^*(T, \mathbb{1})$  has an amenable trace. Applying Proposition 2.5 (see also Theorem 1.1 in [7]) we conclude that there exists a Følner sequence of non-zero finite rank projections  $\{P_n\}_n$ , i.e., we have

$$\lim_{n \rightarrow \infty} \frac{\|[T, P_n]\|_2}{\|P_n\|_2} = 0.$$

(Note that  $P_n$  is not necessarily a proper Følner sequence in the sense of Definition 2.1.) Two cases may appear: if  $\dim P_n \mathcal{H} \leq m$  for some  $m \in \mathbb{N}$ , then choose a subsequence with constant rank and by Proposition 2.8 we conclude that  $T$  has a proper Følner sequence. If the dimensions of  $P_n \mathcal{H}$  are not bounded, then from Proposition 2.2 (iii) we also have that  $T$  has a proper Følner sequence.  $\square$

**3.3. Strongly non-Følner operators.** In the present subsection we study the operators with no Følner sequence. For this we introduce the following notion of operator that is far from having a non-trivial finite dimensional reducing subspace.

**Definition 3.11.** Let  $\mathcal{H}$  be an infinite dimensional Hilbert space and  $T$  an operator on  $\mathcal{H}$ . We will say that  $T$  is *strongly non-Følner* if there exists an  $\varepsilon > 0$  such that all projections  $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$  satisfy

$$\frac{\|TP - PT\|_2}{\|P\|_2} \geq \varepsilon.$$

The following result shows the structure of operators with no proper Følner sequence. Its proof is long and technical and we refer to Section 3 in [38] for details.

**Theorem 3.12.** *Let  $T \in \mathcal{L}(\mathcal{H})$  with  $\dim \mathcal{H} = \infty$ . Then  $T$  has no proper Følner sequence if and only if  $T$  has an orthogonal sum representation  $T = T_0 \oplus \tilde{T}$  on  $\mathcal{H} = \mathcal{H}_0 \oplus \tilde{\mathcal{H}}$ , where  $\dim \mathcal{H}_0 < \infty$  and  $\tilde{T}$  is strongly non-Følner.*

Next we mention some concrete examples of strongly non-Følner operators. We will use the amenable trace that appears in Proposition 2.5 as an obstruction. Recall the definition of the

Cuntz algebra  $\mathcal{O}_n$  (cf. [25, 26]): it is the universal  $C^*$ -algebra generated by  $n \geq 2$  non-unitary isometries  $S_1, \dots, S_n$  with the property that their final projections add up to the identity, i.e.,

$$(3.3) \quad \sum_{k=1}^n S_k S_k^* = \mathbb{1}.$$

This condition implies in particular that the range projections are pairwise orthogonal, i.e.,

$$(3.4) \quad S_l^* S_k = \delta_{lk} \mathbb{1}.$$

It is easy to realize the Cuntz algebra on the complex Hilbert space  $\ell_2$  of square summable sequences.

**Proposition 3.13.** *The Cuntz algebra  $\mathcal{O}_n$ ,  $n \geq 2$ , is singly generated and its generator is strongly non-Følner.*

*Proof.* By Corollary 4 (or Theorem 9) in [40] any Cuntz algebra  $\mathcal{O}_n$ ,  $n \geq 2$ , has a single generator  $C_n$ , i.e.,  $\mathcal{O}_n = C^*(C_n)$ . We assert that  $C_n$  is strongly non-Følner. Indeed, assume that, to the contrary, it is not; then by Corollary 3.14 (ii),  $C_n$  is finite. By Corollary 3.9, it would follow that  $\mathcal{O}_n = C^*(C_n)$  has an amenable trace  $\tau$ . But this gives a contradiction since applying  $\tau$  to the equations (3.3) and (3.4) we obtain  $n = 1$ .  $\square$

Other examples of a strongly non-Følner operators can be obtained from the proof of Theorem 5 in [30]. It is also worth mentioning that Corollary 4 in [20] gives an example of a strongly non-Følner operator generating a type  $II_1$  factor.

Theorem 3.10 allows to divide the class of bounded linear operators into the following mutually disjoint subclasses summarized in the following table:

	Operators with a proper Følner sequence	Operators with no proper Følner sequence
Finite block reducible	$\mathcal{W}_{0+}$	$\mathcal{W}_{0-}$
Non finite block reducible	$\mathcal{W}_{1+}$	$\mathcal{S}$

TABLE 1.

Finally, we conclude this analysis with the following immediate consequences:

**Corollary 3.14.** *Let  $T \in \mathcal{L}(\mathcal{H})$ . Then*

- (i)  *$T$  is a finite operator if and only if  $T$  is in one of the following mutually disjoint classes:  $\mathcal{W}_{0+}$ ,  $\mathcal{W}_{0-}$ ,  $\mathcal{W}_{1+}$ .*
- (ii)  *$T$  is not a finite operator (i.e., it is of class  $\mathcal{S}$ ) if and only if  $T$  is strongly non-Følner.*
- (iii) *The class of strongly non-Følner operators is open and dense in  $\mathcal{L}(\mathcal{H})$ .*

*Proof.* The characterization of finite operators and its complement stated in (i) and (ii) follows from Theorem 3.10 and Williams' theorem. To prove part (iii) we use that the class of finite operators is closed and nowhere dense (cf. [35]). Therefore the set of strongly non-Følner operators is an open and dense subset of  $\mathcal{L}(\mathcal{H})$ .  $\square$

As a summary let us mention that proper Følner sequences for operators provide a useful and natural tool to analyze the class of finite operators. To illustrate this with an example note that the preceding corollary already implies that the class of finite operators is closed in  $\mathcal{L}(H)$ .

## 4. FØLNER SEQUENCES IN OPERATOR ALGEBRAS

We start the analysis of Følner sequences in the context of operator algebras stating some approximation results for amenable traces. We will apply them to spectral approximation problems of scalar spectral measures. In the final part of this section we will give an abstract characterization in terms of unital completely positive maps of  $C^*$ -algebras admitting a faithful essential representation which has a Følner sequence or, equivalently, an amenable trace.

**4.1. Approximations of amenable traces.** Part (i) of the following result is a standard weak\*-compactness argument. Part (ii) is known to experts (see, e.g., Exercise 6.2.6 in [17]) or [1] for a complete proof).

**Proposition 4.1.** *Let  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$  be a unital separable  $C^*$ -algebra.*

- (i) *If  $\mathcal{A}$  has a Følner sequence  $\{P_n\}_n$ , then  $\mathcal{A}$  has an amenable trace.*
- (ii) *Assume that  $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$ , and let  $\tau$  be an amenable trace on  $\mathcal{A}$ . Then  $\mathcal{A}$  has a Følner sequence  $\{P_n\}_n$  satisfying*

$$(4.1) \quad \tau(A) = \lim_{n \rightarrow \infty} \frac{\text{Tr}(AP_n)}{\text{Tr}(P_n)}, \quad A \in \mathcal{A},$$

where  $\text{Tr}$  denotes the canonical trace on  $\mathcal{L}(\mathcal{H})$ .

We will now present an application of Proposition 4.1 (ii) to obtain an approximation result for scalar spectral measures. For this we need to recall from [6] the definition of Szegő pairs for a concrete  $C^*$ -algebra  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ . This notion incorporates the good spectral approximation behavior of scalar spectral measures of selfadjoint elements in  $\mathcal{A}$  and is motivated by Szegő's classical approximation results mentioned in the introduction.

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra acting on  $\mathcal{H}$  and let  $\tau$  be a tracial state on  $\mathcal{A}$ . For any selfadjoint element  $T \in \mathcal{A}$  we denote by  $\mu_T$  the spectral measure associated with the trace  $\tau$  of  $\mathcal{A}$ . Consider a sequence  $\{P_n\}_n$  of non-zero finite rank projections on  $\mathcal{H}$  and write the corresponding (selfadjoint) compressions as  $T_n := P_n T P_n$ . Denote by  $\mu_T^n$  the probability measure on  $\mathbb{R}$  supported on the spectrum of  $T_n$ , i.e., for any  $T = T^* \in \mathcal{A}$  we have

$$\mu_T^n(\Delta) := \frac{N_T^n(\Delta)}{\|P_n\|_1}, \quad \Delta \subset \mathbb{R} \text{ Borel},$$

where  $N_T^n(\Delta)$  is the number of eigenvalues of  $T_n$  (multiplicities counted) contained in  $\Delta$ . We say that  $(\{P_n\}_n, \tau)$  is a *Szegő pair* for  $\mathcal{A}$  if  $\mu_T^n \rightarrow \mu_T$  weakly for all selfadjoint elements  $T \in \mathcal{A}$ , i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \left( f(\lambda_{1,n}) + \cdots + f(\lambda_{d_n,n}) \right) = \int f(\lambda) d\mu_T(\lambda), \quad f \in C_0(\mathbb{R}),$$

where  $d_n = \|P_n\|_1$  is the dimension of the  $P_n \mathcal{H}$  and  $\{\lambda_{1,n}, \dots, \lambda_{d_n,n}\}$  are the eigenvalues (repeated according to multiplicity) of  $T_n$ .

By [6, Theorem 6 (i), (ii)], if  $(\{P_n\}_n, \tau)$  is a Szegő pair for  $\mathcal{A}$ , then  $\{P_n\}_n$  must be a Følner sequence for  $\mathcal{A}$ ,  $\tau$  must be an amenable trace, and equation (4.1) must hold for every  $A \in \mathcal{A}$ . Proposition 4.1 (ii) allows one to complete any amenable trace  $\tau$  on  $\mathcal{A}$  with a Følner sequence so that the pair  $(\{P_n\}_n, \tau)$  is a Szegő pair for  $\mathcal{A}$ , as follows. The proof of the following result requires the construction of an increasing sequence of operators that approximates simultaneously the corresponding commutator and the amenable trace. We refer to Theorem 3.2 in [1] for details

**Theorem 4.2.** *Let  $\mathcal{A}$  be a unital, separable  $C^*$ -algebra acting on a separable Hilbert space  $\mathcal{H}$ , and assume that  $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$ . If  $\tau$  is an amenable trace on  $\mathcal{A}$ , then there exists a proper Følner sequence  $\{P_n\}_n$  such that  $(\{P_n\}_n, \tau)$  is a Szegő pair for  $\mathcal{A}$ .*

*Remark 4.3.* We conclude this subsection recalling that an important step in the proof of the Arveson-Bédos spectral approximation results mentioned in the introduction is the compatibility between the choice of the Følner sequence in the Hilbert space and the amenable trace. In fact, if a unital and separable concrete  $C^*$ -algebra  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$  has an amenable trace  $\tau$  and  $\{P_n\}_n$  is a Følner sequence of non-zero finite rank projections for  $\mathcal{A}$  it is needed that the projections approximate the amenable trace in the following natural sense

$$(4.2) \quad \tau(A) = \lim_{n \rightarrow \infty} \frac{\text{Tr}(AP_n)}{\text{Tr}(P_n)}, \quad A \in \mathcal{A}.$$

Now given  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$  with an amenable trace  $\tau$  it is possible to construct a Følner sequence in different ways. As observed by Bédos in [7] one way to obtain a Følner sequence  $\{P_n\}$  for  $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$  is essentially contained in [22, 23]. In these articles Connes adapts the group theoretic methods by Day and Namioka to the context of operators. Using this technique one loses track of the initial amenable trace  $\tau$ , in the sense that the sequence  $\{P_n\}$  does not necessarily satisfy (4.2). To avoid this problem one may assume in addition that  $\mathcal{A}$  has a unique tracial state. This is sufficient to guarantee a good spectral approximation behavior of relevant examples like almost Mathieu operators, which are contained in the irrational rotation algebra (cf. [10]).

In contrast with the previous method, the construction of a Følner sequence given in [41, Theorem 6.1] (see also [17, Theorem 6.2.7]) allows one to approximate the original trace as in Eq. (4.2). In the precedent theorem it was crucial to use this method to prove a spectral approximation result in the spirit of Arveson and Bédos, but removing the hypothesis of a unique trace (compare Theorem 4.2 with [7, Theorem 1.3] or [6, Theorem 6 (iii)] and the formulation in p. 354 of [4]).

**4.2. Følner  $C^*$ -algebras.** The existence of a Følner sequence for a set of operators  $\mathcal{T}$  is a weaker notion than quasidiagonality. Recall that a set of operators  $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$  is said to be quasidiagonal if there exists an increasing sequence of finite-rank projections  $\{P_n\}_{n \in \mathbb{N}}$  converging strongly to  $\mathbb{1}$  and such that

$$(4.3) \quad \lim_n \|TP_n - P_nT\| = 0, \quad T \in \mathcal{T}.$$

(See, e.g., [32, 49] or Chapter 16 in [17].) The existence of proper Følner sequences can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the underlying spaces. It can be easily shown that if  $\{P_n\}_n$  quasidiagonalizes a family of operators  $\mathcal{T}$ , then this sequence of non-zero finite rank orthogonal projections is also a proper Følner sequence for  $\mathcal{T}$ . The unilateral shift is a basic example that shows the difference between the notions of proper Følner sequences and quasidiagonality. It is a well-known fact that the unilateral shift  $S$  is not a quasidiagonal operator. (This was shown by Halmos in [31]; in fact, in this reference it is shown that  $S$  is not even quasitriangular.) In the setting of abstract  $C^*$ -algebras it can also be shown that a  $C^*$ -algebra containing a non-unitary isometry is not quasidiagonal (see, e.g., [14, 17]).

In [50], Voiculescu characterized abstractly quasidiagonality for unital separable  $C^*$ -algebras in terms of unital completely positive (u.c.p.) maps<sup>4</sup> (see also [49]). This has become by now the standard definition of quasidiagonality for operator algebras (see, for example, [17, Definition 7.1.1]):

**Definition 4.4.** A unital separable  $C^*$ -algebra  $\mathcal{A}$  is called *quasidiagonal* if there exists a sequence of u.c.p. maps  $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$  which is both asymptotically multiplicative (i.e.,  $\|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\| \rightarrow 0$  for all  $A, B \in \mathcal{A}$ ) and asymptotically isometric (i.e.,  $\|A\| = \lim_{n \rightarrow \infty} \|\varphi_n(A)\|$  for all  $A \in \mathcal{A}$ ).

<sup>4</sup>Recall that in this context a linear map  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  between unital  $C^*$ -algebras  $\mathcal{A}, \mathcal{B}$  is called *unital completely positive (u.c.p.)*, if  $\varphi(\mathbb{1}) = \mathbb{1}$  and if the inflations  $\varphi_n := \varphi \otimes \text{id}_n: \mathcal{A} \otimes M_n(\mathbb{C}) \rightarrow \mathcal{B} \otimes M_n(\mathbb{C})$  are positive for all  $n \geq 1$ .

Inspired by Voiculescu's work on quasidiagonality we introduce in this section an abstract definition of a Følner  $C^*$ -algebra and formulate our main result characterizing Følner  $C^*$ -algebras in terms of Følner sequences and also of amenable traces.

Recall that  $\text{tr}(\cdot)$  denotes the unique tracial state on a matrix algebra  $M_n(\mathbb{C})$ .

**Definition 4.5.** Let  $\mathcal{A}$  be a unital, separable  $C^*$ -algebra.

- (i) We say that  $\mathcal{A}$  is a *Følner  $C^*$ -algebra* if there exists a sequence of u.c.p. maps  $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$  such that

$$(4.4) \quad \lim_n \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}} = 0, \quad A, B \in \mathcal{A},$$

where  $\|F\|_{2,\text{tr}} := \sqrt{\text{tr}(F^*F)}$ ,  $F \in M_n(\mathbb{C})$ .

- (ii) We say that  $\mathcal{A}$  is a *proper Følner  $C^*$ -algebra* if there exists a sequence of u.c.p. maps  $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$  satisfying (4.4) and which, in addition, are asymptotically isometric, i.e.,

$$(4.5) \quad \|A\| = \lim_n \|\varphi_n(A)\|, \quad A \in \mathcal{A}.$$

It is clear that if  $\mathcal{A}$  is a separable, unital and quasidiagonal  $C^*$ -algebra (cf. Definition 4.4), then  $\mathcal{A}$  is a proper Følner algebra. The Toeplitz algebra serves as a counter-example to the reverse implication.

Although, in principle, the two concepts—Følner and proper Følner—seem to be different for  $C^*$ -algebras, we can show that they indeed define the same class of unital, separable  $C^*$ -algebras. The proof of the next proposition includes a useful trick so that we will include it here (cf. [1, Proposition 3.2]).

**Proposition 4.6.** *Let  $\mathcal{A}$  be a unital separable  $C^*$ -algebra. Then  $\mathcal{A}$  is a Følner  $C^*$ -algebra if and only if  $\mathcal{A}$  is a proper Følner  $C^*$ -algebra.*

*Proof.* Assume that  $\mathcal{A}$  is a Følner  $C^*$ -algebra, and let  $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$  be a sequence of u.c.p. maps such that (4.4) holds. Considering the direct sum of a sufficiently large number of copies of  $\varphi_n$ , for each  $n$ , we may assume that

$$(4.6) \quad \lim_{n \rightarrow \infty} \frac{n}{k(n)} = 0.$$

Let  $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  be a faithful representation of  $\mathcal{A}$  on a separable Hilbert space  $\mathcal{H}$ . Let  $\{P_n\}_n$  be an increasing sequence of orthogonal projections on  $\mathcal{H}$ , converging to  $\mathbb{1}$  in the strong operator topology and such that  $\dim P_n(\mathcal{H}) = n$  for all  $n$ . Then for all  $A \in \mathcal{A}$  we have  $\|A\| = \lim_n \|P_n \pi(A) P_n\|$ . Let  $\psi_n: \mathcal{A} \rightarrow M_{k(n)+n}(\mathbb{C})$  be given by:

$$\psi_n(A) = \varphi_n(A) \oplus P_n \pi(A) P_n, \quad A \in \mathcal{A}.$$

Then  $\psi_n$  is a u.c.p. map. For  $A, B \in \mathcal{A}$ , set  $X_n = P_n \pi(A) (1 - P_n) \pi(B) P_n$ . Then we have

$$\begin{aligned} \|\psi_n(AB) - \psi_n(A)\psi_n(B)\|_{2,\text{tr}}^2 &\leq \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}}^2 + \frac{\text{Tr}(X_n^* X_n)}{k(n) + n} \\ &\leq \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}}^2 + \frac{n\|A\|^2\|B\|^2}{k(n) + n}. \end{aligned}$$

Using (4.6) we get

$$\lim_n \|\psi_n(AB) - \psi_n(A)\psi_n(B)\|_{2,\text{tr}} = 0.$$

On the other hand, for  $A \in \mathcal{A}$ , we have

$$\|A\| - \|\psi_n(A)\| \leq \|A\| - \|P_n \pi(A) P_n\| \rightarrow 0$$

so that (4.5) holds for the sequence  $(\psi_n)$ . This concludes the proof.  $\square$

For the next result recall that a representation  $\pi$  of an abstract C\*-algebra  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$  is called *essential* if  $\pi(\mathcal{A})$  contains no nonzero compact operators. The proof uses the same approximation technique as the proof of Theorem 4.2 (see Theorem 3.4 in [1] for details).

**Theorem 4.7.** *Let  $\mathcal{A}$  be a unital separable C\*-algebra. Then the following conditions are equivalent:*

- (i) *There exists a faithful representation  $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  such that  $\pi(\mathcal{A})$  has a Følner sequence.*
- (ii) *There exists a faithful essential representation  $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  such that  $\pi(\mathcal{A})$  has a Følner sequence.*
- (iii) *Every faithful essential representation  $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  satisfies that  $\pi(\mathcal{A})$  has a proper Følner sequence.*
- (iv) *There exists a non-zero representation  $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  such that  $\pi(\mathcal{A})$  has an amenable trace.*
- (v) *Every faithful representation  $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  satisfies that  $\pi(\mathcal{A})$  has an amenable trace.*
- (vi)  *$\mathcal{A}$  is a Følner C\*-algebra.*

**Remark 4.8.** (i) The class of C\*-algebras introduced in this section has been considered before by Bédos. In [7] the author defines a C\*-algebra  $\mathcal{A}$  to be *weakly hypertracial* if  $\mathcal{A}$  has a non-degenerate representation  $\pi$  such that  $\pi(\mathcal{A})$  has a hypertrace. In this sense, the preceding theorem gives a new characterization of weakly hypertracial C\*-algebras in terms of u.c.p. maps.

- (ii) Note also that the equivalences between (i), (iv) and (v) in Theorem 4.7 are basically known (see [7]).

We conclude mentioning that in the study of growth properties of C\*-algebras (and motivated by previous work done by Arveson and Bédos) Vaillant defined the following natural unital C\*-algebra (see Section 3 in [46]): given an increasing sequence  $\mathcal{P} := \{P_n\}_n \subset \mathcal{P}_{\text{fin}}(\mathcal{H})$  of orthogonal finite rank projections strongly converging to  $\mathbb{1}$ , consider the set of all bounded linear operators in  $\mathcal{H}$  that have  $\mathcal{P}$  as a proper Følner sequence, i.e.,

$$\mathcal{F}_{\mathcal{P}}(\mathcal{H}) := \left\{ X \in \mathcal{L}(\mathcal{H}) \mid \lim_{n \rightarrow \infty} \frac{\|XP_n - P_nX\|_2}{\|P_n\|_2} = 0 \right\}.$$

This unital C\*-subalgebra of  $\mathcal{L}(\mathcal{H})$  (called Følner algebra by Hagen, Roch and Silbermann in Section 7.2.1 of [29]) has shown to be very useful in the analysis of the classical Szegő limit theorems for Toeplitz operators and some generalizations of them (see, e.g., Section 7.2 of [29] and [12]).

The C\*-algebra  $\mathcal{F}_{\mathcal{P}}$  is always non-separable for the operator norm. Indeed, consider the  $\ell^\infty$ -direct sum of matrix algebras  $\mathcal{A} = \prod_i M_{n_i}(\mathbb{C})$ , where  $n_i$  are the ranks of the orthogonal projections  $P_{i+1} - P_i$ ,  $i \in \mathbb{N}$ , with norm given by  $\|(a_i)\| = \sup_i \|a_i\|$ . It is clear that  $\mathcal{A}$  is not separable, and the elements of  $\mathcal{A}$  can be seen inside  $\mathcal{F}_{\mathcal{P}}$  as block-diagonal operators, so the algebra  $\mathcal{F}_{\mathcal{P}}$  is also non-separable.

## 5. FINAL REMARKS: FØLNER VERSUS PROPER FØLNER

As was mentioned at the beginning of Section 2, Følner sequences appeared first in the context of groups. Note that if countable discrete group  $\Gamma$  has a Følner sequence one can always find another Følner sequence which, in addition to Eq. (2.1), is also proper, i.e.,  $\Gamma_i \subset \Gamma_j$  if  $i \leq j$  and  $\Gamma = \cup_i \Gamma_i$ . In the context of operators and due to the linear structure of the underlying Hilbert spaces the difference between Følner sequence and proper Følner sequence is relevant. As was mentioned after Proposition 2.7 if  $T = T_0 \oplus T_1$  is a finite block reducible operator on the Hilbert space  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ , with  $1 \leq \dim \mathcal{H}_0 < \infty$ , and  $T_1$  strongly non-Følner (cf. Subsection 3.3), then  $T$  has an obvious constant Følner sequence but can not have

a proper Følner sequence. Moreover, Proposition 2.8 shows that the difference between Følner and proper Følner sequence for single operators can only appear in the case when there is a non-trivial finite dimensional invariant subspace.

At the level of abstract  $C^*$ -algebras Proposition 4.6 shows that Følner  $C^*$ -algebras and proper Følner  $C^*$ -algebras define the same class of unital separable  $C^*$ -algebras. Note that by Theorem 4.7 (i) the direct sum of a matrix algebra and the Cuntz algebra

$$\mathcal{A} := M_n(\mathbb{C}) \oplus \mathcal{O}_n$$

is a Følner (hence proper Følner)  $C^*$ -algebra. But in its natural representation on  $\mathcal{H} := \mathbb{C}^n \oplus \ell_2$  this algebra can not have a proper Følner sequence because the representation is not essential (see Theorem 4.7 (iii)).

Finally, if  $\mathcal{B}$  is a unital  $C^*$ -subalgebra of a Følner  $C^*$ -algebra  $\mathcal{A}$ , then one can restrict the u.c.p. maps of  $\mathcal{A}$  to  $\mathcal{B}$  to show that  $\mathcal{B}$  is also a Følner  $C^*$ -algebra. This is not true if  $\mathcal{B}$  is a non-unital  $C^*$ -subalgebra (i.e., if  $1_{\mathcal{A}} \notin \mathcal{B}$ ). Consider, for example, the concrete  $C^*$ -algebra on  $\mathcal{K} := \ell_2 \oplus \mathcal{H}$  given by

$$\mathcal{A} := C^*(S) \oplus C^*(T_1),$$

where  $S$  is the unilateral shift and  $T_1$  is a strongly non-Følner operator. Then, again,  $\mathcal{A}$  is a Følner (hence proper Følner)  $C^*$ -algebra, but the non-unital  $C^*$ -subalgebra  $\mathcal{B} := 0 \oplus C^*(T_1)$  is not a Følner  $C^*$ -algebra.

**Acknowledgment.** The two first-named authors would like to acknowledge the *Instituto Superior Técnico, Lisboa*, for the invitation and hospitality during the *Workshop on Operator Theory and Operator Algebras 2012*.

## REFERENCES

- [1] P. Ara and F. Lledó, *Amenable traces and Følner  $C^*$ -algebras*, preprint 2013, arXiv:math.OA/1206.1488v3.
- [2] M. Ahues, A. Largillier and B.V. Limaye, *Spectral Computations for Bounded Operators*, Chapman & Hall/CRC, Boca Raton, 2001.
- [3] W. Arveson, *Improper filtrations for  $C^*$ -algebras: spectra of unilateral tridiagonal operators*, Acta Sci. Math. (Szeged) **57** (1993), 11-24.
- [4] W. Arveson,  *$C^*$ -algebras and numerical linear algebra*, J. Funct. Anal. **122** (1994), 333-360.
- [5] W. Arveson, *The role of  $C^*$ -algebras in infinite dimensional numerical linear algebra*, In  *$C^*$ -Algebras: 1943-1993. A Fifty Year Celebration*, R.S. Doran (ed.), Contemporary Mathematics Vol. 167, American Mathematical Society, Providence, Rhode Island, 1994; pp. 115-129.
- [6] E. Bédos, *On Følner nets, Szegő's theorem and other eigenvalue distribution theorems*, Expo. Math. **15** (1997), 193 – 228. Erratum: Expo. Math. **15** (1997), 384.
- [7] E. Bédos, *Notes on hypertraces and  $C^*$ -algebras*, J. Operator Theory **34** (1995), 285-306.
- [8] E. Bédos, *On filtrations for  $C^*$ -algebras*, Houston J. Math. **20** (1994), 63-74.
- [9] C.A. Berger, B.I. Shaw, *Selfcommutators of multicyclic hyponormal operators are always trace class*, Bull. Amer. Math. Soc. **79** (1973), 1193-1199.
- [10] F.P. Boca, *Rotation  $C^*$ -algebras and Almost Mathieu Operators*, The Theta Foundation, Bucarest, 2001.
- [11] A. Böttcher,  *$C^*$ -algebras in numerical analysis*, Irish Math. Soc. Bull. **45** (2000), 57-133.
- [12] A. Böttcher, P. Otte, *The first Szegő limit theorem for non-selfadjoint operators in the Følner algebra*, Math. Scand. **97** (2005), 115-126.
- [13] N.P. Brown, *Invariant means and finite representation theory of  $C^*$ -algebras*, Mem. Am. Math. Soc. **184** (2006) no. 865, 1-105.
- [14] N.P. Brown, *On quasidiagonal  $C^*$ -algebras*, In *Advanced Studies in Pure Mathematics 38: Operator Algebras and Applications*, H. Kosaki (ed.), Mathematical Society of Japan, 2004, pp. 19-64.
- [15] N.P. Brown, *Herrero's approximation problem for quasidiagonal operators*, J. Funct. Anal. **186** (2001), 360-365.
- [16] N.P. Brown, *AF embeddings and the numerical computation of spectra in the irrational rotation algebra*, Numer. Funct. Anal. Optim. **27** (2006), 517-528.
- [17] N.P. Brown and N. Ozawa,  *$C^*$ -Algebras and Finite-Dimensional Approximations*, American Mathematical Society, Providence, Rhode Island, 2008.
- [18] L.G. Brown, R.G. Douglas and P.A. Fillmore, *Extension of  $C^*$ -algebras, operators with compact self-commutators, and  $K$ -homology*, Bull. Amer. Math. Soc. **79** (1973), 973-978.



- [19] L.G. Brown, R.G. Douglas and P.A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of  $C^*$ -algebras*, In *Proceedings of a Conference on Operator Theory*, P.A. Fillmore (ed.), Lecture Notes in Mathematics **345**, Springer Verlag, Berlin, 1973; pp. 58-128.
- [20] J.W. Bunce, *Finite operators and amenable  $C^*$ -algebras*, Proc. Amer. Math. Soc., **56** (1976), 145–151.
- [21] F. Chatelin, *Spectral Approximation of Linear Operators*, Academic Press, New York, 1983.
- [22] A. Connes, *Classification of injective factors. Cases  $II_1$ ,  $II_\infty$ ,  $III_\lambda$ ,  $\lambda \neq 1$* , Ann. Math. **104** (1976), 73-115.
- [23] A. Connes, *On the classification of von Neumann algebras and their automorphisms*, In *Symposia Mathematica*, Vol. XX (*Convegno sulle Algebre  $C^*$  e loro Applicazioni in Fisica Teorica, Convegno sulla Teoria degli Operatori Indice e Teoria  $K$* , INDAM, Rome, 1975), Academic Press, London, 1976.
- [24] J.B. Conway, *The Theory of Subnormal Operators*, American Mathematical Society, Providence, Rhode Island, 1991.
- [25] J. Cuntz, *Simple  $C^*$ -algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173–185.
- [26] K.R. Davidson,  *$C^*$ -Algebras by Example*, American Mathematical Society, Providence, Rhode Island, 1996.
- [27] K.R. Davidson, *Essentially normal operators. A glimpse at Hilbert space operators*, Oper. Theory Adv. Appl., **207** (2010), 209–222.
- [28] P.A. Fillmore, J.G. Stampfli, J.P. Williams, *On the essential numerical range, the essential spectrum, and a problem of Halmos*, Acta Sci. Math. (Szeged) **33** (1972), 179-192.
- [29] R. Hagen, S. Roch and B. Silbermann,  *$C^*$ -Algebras and Numerical Analysis*, Marcel Dekker, Inc., New York, 2001.
- [30] P.R. Halmos, *Commutators of operators, II*, Amer. J. Math. **76** (1954), 129-136.
- [31] P.R. Halmos, *Quasitriangular operators*, Acta Sci. Math. (Szeged) **29** (1968), 283-293.
- [32] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Society **76** (1970), 887-933.
- [33] A.C. Hansen, *On the approximation of spectra of linear operators on Hilbert spaces*, J. Funct. Anal. **254** (2008), 2092-2126.
- [34] A.C. Hansen, *On the Solvability Complexity Index, the  $n$ -pseudospectrum and approximations of spectra of operators*, J. Amer. Math. Soc. **24** (2011), 81-124.
- [35] D. Herrero, *The set of finite operators is nowhere dense*, Canad. Math. Bull. **32** (1989), 320-326.
- [36] D. Herrero, *What is a finite operator?*, In *Linear and Complex Analysis Problem Book*, V.P. Havin and N.K. Nikolski (eds.), Springer Verlag, Berlin, 1994. pp. 226-228.
- [37] F. Lledó, *On spectral approximation, Følner sequences and crossed products*, J. Approx. Theory 2012, doi: 10.1016/j.jat.2012.10.003.
- [38] F. Lledó and D. Yakubovich, *Følner sequences and finite operators*, J. Math. Anal. Appl., **403** (2013) 464-476. doi: 10.1016/j.jmaa.2013.02.037
- [39] J.v. Neumann, *Zur Algebra der Funktionaloperationen und der Theorie der normalen Operatoren* **102** (1929), 307–427.
- [40] C.L. Olsen and W.R. Zame, *Some  $C^*$ -algebras with a single generator*, Trans. Amer. Math. Soc. **215** (1976), 205-217.
- [41] N. Ozawa, *About the QWEP conjecture*, Internat. J. Math. **15** (2004), 501–530.
- [42] A.L. Paterson, *Amenability*, American Mathematical Society, Providence, Rhode Island, 1988.
- [43] S. Roch and B. Silbermann, *Szegő limit theorems for operators with almost periodic diagonals*, Operators and Matrices **1** (2007) 1-29.
- [44] M. A. Shubin, *Pseudodifferential operators and spectral theory*. Second edition. Springer Verlag, Berlin, 2001.
- [45] P. Tilli, *Some Results on Complex Toeplitz Eigenvalues*, J. Math. Anal. Appl. **239** (1999), 390-401.
- [46] G. Vaillant, *Følner conditions, nuclearity, and subexponential growth in  $C^*$ -algebras*, J. Funct. Anal. **141** (1996), 435-448.
- [47] A. Vershik, *Amenability and approximation of infinite groups*, Sel. Math. Sov. **2** (1982), 311-330.
- [48] D. Voiculescu, *A non-commutative Weyl von-Neumann theorem*, Rev. Roum. Math. Pures et Appl. **XXI** (1976), 97-113.
- [49] D. Voiculescu, *Around quasidiagonal operators*, Integr. Equat. Oper. Theory **17** (1993), 137-149.
- [50] D. Voiculescu, *A note on quasi-diagonal  $C^*$ -algebras and homotopy*, Duke Math. J. **62** (1991), 267–271.
- [51] A.L. Vol'berg, V.V. Peller and D.V. Yakubovich, *A brief excursion to the theory of hyponormal operators*, Leningrad Math. J. **2** (1991), 211-243.
- [52] H. Widom, *Eigenvalue distribution for nonselfadjoint Toeplitz matrices*, Oper. Theor. Adv. Appl. **71** (1994), 1-7.
- [53] J.P. Williams, *Finite operators*, Proc. Amer. Math. Soc. **26** (1970), 129-136.

DEPARTMENT OF MATHEMATICS, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA (BARCELONA), SPAIN

*E-mail address:* para@mat.uab.cat

DEPARTMENT OF MATHEMATICS, UNIVERSITY CARLOS III MADRID, AVDA. DE LA UNIVERSIDAD 30, E-28911 LEGANÉS (MADRID), SPAIN AND INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC - UAM - UC3M - UCM)

*E-mail address:* flledo@math.uc3m.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTONOMA DE MADRID, CANTOBLANCO 28049 (MADRID) SPAIN AND INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC - UAM - UC3M - UCM)

*E-mail address:* dmitry.yakubovich@uam.es